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ON CERTAIN DETERMINANTS ASSOCIATED WITH TRANSFORMATIONS EMPLOYED IN THERMODYNAMICS.

BY J. E. TREVOR.

1. Introduction. When a fluid mixture of two distinct component substances is in a state of thermodynamic equilibrium, the energy $E(x_1, x_2, x_3, x_4)$ of the mixture is a positively homogeneous function, of degree one, of the volume x_1, the entropy x_2, and the component-masses x_3, x_4 of the mixture. If δE denotes the increment of E when the variables are given independent increments $\delta x_1, \cdots, \delta x_4$, it is concluded from the principles of thermodynamics that the equilibrium is stable when the sum of terms of order two in Taylor's expansion of δE is positive for all sets of sufficiently small increments that do not satisfy the conditions

$$\delta x_1/x_1 = \delta x_2/x_2 = \delta x_3/x_3 = \delta x_4/x_4.$$

Again, when the specific volume y_1, the specific entropy y_2, and the specific component-masses y_3, y_4 are defined by equations $y_i = x_i/(x_3 + x_4)$, the energy E is a function $(x_3 + x_4)\cdot E(y_1, y_2, y_3, 1 - y_3)$, i.e. a function $(x_3 + x_4)\cdot e(y_1, y_2, y_3)$; whereupon it is concluded[*] that stability is ensured when the sum of terms of order two in the expansion of δe is positive for all sufficiently small increments of y_1, y_2, y_3. From either of these two criteria of stability a set of necessary and sufficient conditions of stability may be deduced.

To obtain equivalent sets of conditions in other independent variables it is customary, on writing p_i for $\partial E/\partial x_i$, to employ the transformations

$$x_1' = p_1, \quad x_2' = x_2, \quad x_3' = x_3, \quad x_4' = x_4, \quad E' = E - p_1x_1;$$

$$x_1' = x_1, \quad x_2' = p_2, \quad x_3' = x_3, \quad x_4' = x_4, \quad E' = E - p_2x_2;$$

$$x_1' = p_1, \quad x_2' = p_2, \quad x_3' = x_3, \quad x_4' = x_4, \quad E' = E - p_1x_1 - p_2x;_2$$

and similar transformations with reference to y_1, y_2, y_3, e. The conditions obtained, and their immediate consequences, are inequalities giving the signs of the hessians of the functions E, E', e, and e', and of the principal minors of these hessians.

Now the elements of any one of these determinants include derivatives with regard to a of the variables p_i, where a may be zero. In seeking a convenient rule for finding the sign of any determinant D of the set, I have observed that D is positive when a is even (0, 2, or 4), and is negative when a is odd; save that D vanishes identically when it is the determinant of the second derivatives, of a function E or E', with regard to all the x's that occur in the function. The exception asserts that when any of the p_i are held constant the others are not independent; which is obviously true, since the first derivatives of a homogeneous function of degree one are connected by a relation.

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[*] J. E. Trevor, Amer. Math. Monthly, vol. 26, 444 (1919).

The observation that the signs of the determinants D are given by a simple rule has led me to seek a theorem of which the rule is a manifestation. The application of the result found, unlike that of the empirical rule, is restricted neither to a limited number of variables, to analytic functions, to positive values of the variables, nor to *principal* minors.

2. The Minors of the Hessians of Certain Related Functions. Let $e(x_1, x_2, \cdots, x_n)$ be a continuous function of the independent variables $x_1, x_2, \cdots, x_n$, with continuous first and second derivatives. Writing $p_i = \partial e/\partial x_i$, consider the transformations

$$x_1' = p_1, \; \cdots, \; x_m' = p_m, \qquad x_{m+1}' = x_{m+1}, \; \cdots, \; x_n' = x_n,$$

$$f_m = e - \sum_{i=1}^{m} p_i x_i \qquad (m = 1, 2, \cdots, n).$$

The jacobians of these transformations are the hessian of e and the principal minors of this hessian. When any of these jacobians vanish, the corresponding transformation is degenerate and shall be excluded from the set. The differentials of the functions $f_m(p_1, \cdots, p_m, x_{m+1}, \cdots, x_n)$ are

$$df_m = -\sum_{i=1}^{m} x_i dp_i + \sum_{j=m+1}^{n} p_j dx_j.$$

Let us now consider the set of $2n$ elements, in "normal order,"

$$p_1x_1 \quad p_2x_2 \quad \cdots \quad p_nx_n;$$

and from this set form an arbitrary combination C, in "normal order," by suppressing any n elements. As an illustration let us choose the set for which $m = 4$ and $n = 9$, and from it form the combination

$$p_1x_1 \quad {*x_2} \quad p_3{*} \quad p_4x_4 \quad {**} \quad p_6{*} \quad {*x_7} \quad p_8{*} \quad {**},$$

where the suppressions are indicated by asterisks. Terming the letters of the sets $x_1, \cdots, x_m$ and $p_1, \cdots, p_m$ respectively the x_{I} and the p_{I}, and terming the remaining x's and p's respectively the x_{II} and the p_{II}, we proceed to tabulate the indices of the elements of each set "present in" and "absent from" the combination C.

	p_1x_1	$^*x_2$	$p_3{}^*$	p_4x_4		$**$	$p_6{}^*$	$^*x_7$	$p_8{}^*$	$**$
Present x_{I}	1	2		4	Present x_{II}			7		
Absent x_{I}			3		Absent x_{II}	5	6		8	9
Present p_{I}	1		3	4	Present p_{II}		6		8	
Absent p_{I}		2			Absent p_{II}	5		7		9

On forming the jacobian of the letters of C, with regard to the variables x_{I}, x_{II}, hereby denoting $\partial^2 e/\partial x_i \partial x_j$ by e_{ij}; and on forming the jacobian of these letters with regard to the variables p_{I}, x_{II}, hereby denoting the second derivative of f_m, i.e. of f_4, with regard to its ith and jth variables by f_{ij}; we obtain,

$$\begin{vmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & e_{37} & e_{38} & e_{39} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} & e_{46} & e_{47} & e_{48} & e_{49} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ e_{61} & e_{62} & e_{63} & e_{64} & e_{65} & e_{66} & e_{67} & e_{68} & e_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ e_{81} & e_{82} & e_{83} & e_{84} & e_{85} & e_{86} & e_{87} & e_{88} & e_{89} \end{vmatrix}, \qquad \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_{11} & -f_{12} & -f_{13} & -f_{14} & -f_{15} & -f_{16} & -f_{17} & -f_{18} & -f_{19} \\ -f_{21} & -f_{22} & -f_{23} & -f_{24} & -f_{25} & -f_{26} & -f_{27} & -f_{28} & -f_{29} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -f_{41} & -f_{42} & -f_{43} & -f_{44} & -f_{45} & -f_{46} & -f_{47} & -f_{48} & -f_{49} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} & f_{67} & f_{68} & f_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ f_{81} & f_{82} & f_{83} & f_{84} & f_{85} & f_{86} & f_{87} & f_{88} & f_{89} \end{vmatrix}.$$

By inspection we observe that the first of these jacobians (apart from its sign) is obtainable from the hessian of $e(x_{\text{I}}, x_{\text{II}})$ by deleting rows and columns as follows.

Illustrative Case		In the General Case
Delete columns	124, 7	Delete columns of the present x_{I} and x_{II}.
Retain rows	134, 68	Retain rows of the present p_{I} and p_{II};
I.e., delete rows	2, 579	I.e., delete rows of the absent p_{I} and p_{II}.

We observe similarly that the second jacobian (apart from its sign) is obtainable from the hessian of $f_m(p_{\text{I}}, x_{\text{II}})$ by deleting rows and columns as follows.

Illustrative Case		In the General Case
Delete columns	134, 7	Delete columns of the present p_{I} and x_{II}.
Retain rows	124, 68	Retain rows of the present x_{I} and p_{II};
I.e., delete rows	3, 579	I.e., delete rows of the absent x_{I} and p_{II}.

If we denote the minor of the hessian of a function u, obtained by deleting the rows $i \cdots k$ and the columns $j \cdots l$, by the symbol $U\left(\begin{smallmatrix} i \cdots k \\ j \cdots l \end{smallmatrix}\right)$, the above jacobians are denoted by

$$\sigma_0 \cdot E\begin{pmatrix} 2, & 579 \\ 124, & 7 \end{pmatrix}, \qquad \sigma_1 \cdot F_4\begin{pmatrix} 3, & 579 \\ 134, & 7 \end{pmatrix};$$

where the sign-factors σ_0, σ_1 are yet undetermined. From the above tabulations, the rule for the formulation of the jacobians of the n letters of any combination C, for any values of n and m, is expressed by

$$\sigma_0 \cdot E\begin{pmatrix}\text{Indices of the absent } p_{\text{I}} \text{ and } p_{\text{II}} \\ \text{Indices of the present } x_{\text{I}} \text{ and } x_{\text{II}}\end{pmatrix},$$

and

$$\sigma_1 \cdot F_m\begin{pmatrix}\text{Indices of the absent } x_{\text{I}} \text{ and } p_{\text{II}} \\ \text{Indices of the present } p_{\text{I}} \text{ and } x_{\text{II}}\end{pmatrix};$$

or, more generally by

$$\sigma \cdot U\begin{pmatrix}\text{Indices of the first derivatives of } u \text{ absent from } C \\ \text{Indices of the independent variables of } u \text{ present in } C\end{pmatrix} \quad (u = e, f_m).$$

In seeking the sign-factor σ_0, we consecutively number the places of the n letters in the illustrative combination C,

$$\begin{array}{lccccccccc} \text{Combination,} & p_1 & x_1 & x_2 & p_3 & p_4 & x_4 & p_6 & x_7 & p_8 \\ \text{Place-numbers,} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

and count the number λ_0 of transpositions of letters necessary to bring each of the variables x_i that is present in C to the ith place in C, without disturbing the order of the p_i that occur in C. These transpositions bring the letters of C into the arrangement

$$\begin{array}{ccccccccc} x_1 & x_2 & p_1 & x_4 & p_3 & p_4 & x_7 & p_6 & p_8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

The corresponding transpositions of rows, in the jacobian of the letters of C with regard to the variables x_{I}, x_{II}, bring all the elements 1 of the array on to the principal diagonal. Hence the sign of the jacobian is that of $\sigma_0 = (-1)^{\lambda_0}$. The number λ_0 can indeed have different values. But these are either all even or all odd.

In seeking the sign-factor σ_1, we count the number λ_1 of transpositions necessary to bring each of the variables y_i that is present in C, where $y_i = p_{\text{I}}$, x_{II}, to the ith place in C, without disturbing the order of the x_{I} and p_{II} that occur in C. These transpositions bring the letters of C into the arrangement

$$\begin{array}{ccccccccc} p_1 & x_1 & p_3 & p_4 & x_2 & x_4 & x_7 & p_6 & p_8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

The corresponding transpositions of rows, in the jacobian of the letters of C with regard to the variables p_{I}, x_{II}, bring all the elements 1 of the array on to the principal diagonal. Hence the sign of the jacobian of the letters of the combination is that of $(-1)^{\lambda_1}$. Now the jacobian of the letters of C, with regard to the variables p_{I}, x_{II}, contains the $\mu = 3$ rows (124) of negative elements corresponding to the μ x_{I}'s that are present in C. On factoring out -1 from each of these rows, the determinant to which the jacobian reduces (on deleting the rows and columns of the elements 1, and on taking out the factors -1) is multiplied by

$$\sigma_1 = (-1)^{\lambda_1+\mu}.$$

So, in general, the jacobian of the letters of the combination C, in "normal order," of any n letters of the set

$$p_1x_1 \quad p_2x_2 \quad \cdots \quad p_nx_n,$$

with regard to the variables x_{I}, x_{II}, is a minor of the hessian of $e(x_{\text{I}}, x_{\text{II}})$,

$$(-1)^{\lambda}\cdot E\begin{pmatrix}\text{Indices of the first derivatives of } e \text{ absent from } C \\ \text{Indices of the independent variables of } e \text{ present in } C\end{pmatrix};$$

and the jacobian of the letters of the combination, with regard to the variables p_{I}, x_{II}, is a minor of the hessian of $f_m(p_{\text{I}}, x_{\text{II}})$,

$$(-1)^{\lambda+\mu}\cdot F_m\begin{pmatrix}\text{Indices of the first derivatives of } f_m \text{ absent from } C \\ \text{Indices of the independent variables of } f_m \text{ present in } C\end{pmatrix}.$$

In each of these formulations the numbers λ and μ are counted with reference to the combination and to the variables taken independent; λ being the number of transpositions necessary to bring each of the independent variables to the place of its index in C, and μ being the number of the variables x_{I} present in C.

3. A Matrix of the Hessians and their Minors. It is now proposed to arrange in a column the combinations C, in "normal order," of the elements

$$p_1x_1 \quad p_2x_2 \quad \cdots \quad p_nx_n,$$

taken n at a time; and then to tabulate the jacobians of the letters of each combination, with regard to the variables of the successive sets

$$x_1, x_2, x_3, \cdots, x_n, \qquad p_1, x_2, x_3, \cdots, x_n, \qquad p_1, p_2, x_3, \cdots, x_n,$$
$$\cdots, \qquad p_1, p_2, p_3, \cdots, p_n;$$

thereby expressing the jacobians as minors, of all orders from zero to n, of the respective hessians $E()$, $F_1()$, $\cdots$, $F_n()$.

In this tabulation the successive places of the row corresponding to any given combination will be occupied by elements

$$(-1)^{\lambda}\cdot E(\vdots\vdots\vdots), \qquad (-1)^{\lambda+\mu}\cdot F_m(\vdots\vdots\vdots), \qquad (m = 1, 2, \cdots, n)$$

where λ, μ are counted in each case with reference to the combination and to the variables taken independent. The tabulated array of minors has $(2n)!/(n!)^2$ rows, one for each combination C; and it has $n + 1$ columns, one for each of the successive sets of variables. Hence the number of the elements constituting the array is $(2n)!(n + 1)/(n!)^2$. For $n = 1$ this number is 4, for $n = 2$ it is 18, for $n = 3$ it is 80, and for $n = 4$ it is 350.

Let the ith element $(-1)^\lambda \cdot E(:::)$ of the first column of the array be denoted by E_i. Then, because of the relation

$$\frac{\partial(C_i \quad)}{\partial(x_{\mathrm{I}}, x_{\mathrm{II}})} = \frac{\partial(C_i \quad)}{\partial(p_{\mathrm{I}}, x_{\mathrm{II}})} \frac{\partial(p_{\mathrm{I}}, x_{\mathrm{II}})}{\partial(x_{\mathrm{I}}, x_{\mathrm{II}})} \cdot$$

where the symbol C_i represents the letters of the ith combination, the $(m + 1)$th element of the ith row satisfies the equation

$$E_i = (-1)^{\lambda+\mu} \cdot F_m(:::) \times E\begin{pmatrix} m+1, \cdots, n \\ m+1, \cdots, n \end{pmatrix},$$

or

$$(-1)^{\lambda+\mu} \cdot F_m(:::) = \left[E\begin{pmatrix} m+1, \cdots, n \\ m+1, \cdots, n \end{pmatrix}\right]^{-1} \cdot E_i = a_m E_i.$$

Thus each element of the $(m + 1)$th column of the array differs from the element E_i of the same row by the same factor a_m. We find that the array forms a matrix such that all determinants of orders greater than one that can be formed from it are equal to zero. The matrix is of rank one. The factors a_m are the reciprocals of the hessian $E()$ and of the principal minors obtained by successively deleting the last $1, 2, \cdots, n - 1$ rows and columns of $E()$. The factor a_1 is the reciprocal of $\partial^2 e/\partial x_1^2$. The conclusion that the hessians of the functions e and f_m, and the minors of these hessians, are connected in this way is the theorem sought.

To find the combination C that corresponds to any minor

$$\pm U\begin{pmatrix} \text{Indices of the first derivatives of } u \text{ absent from } C \\ \text{Indices of the independent variables of } u \text{ present in } C \end{pmatrix} \quad (u = e, f_m),$$

of any order from zero to n, of the hessian of any function e or f_m, we have the rule,

Comparing the symbol for the minor with the list of the first derivatives and of the independent variables of u, write the derivatives not absent and the variables present, and arrange them in normal order.

For example let $F_2\binom{23}{13}$ be given, for $n = 3$. The derivatives are $-x_1$, $-x_2$, p_3, and the variables are p_1, p_2, x_3. The derivative not absent is $-x_1$, and the variables present are p_1, x_3. Hence $C = p_1x_1x_3$. Again, let $F_3\binom{2}{2}$ be given, for $n = 4$. Derivatives are $-x_1$, $-x_2$, $-x_3$, p_4, and variables are p_1, p_2, p_3, x_4. Derivatives not absent are $-x_1$, $-x_3$, p_4, and the variable present is p_2. Hence $C = x_1p_2x_3p_4$.

In the case of a principal minor, which is formed by deleting the same columns as rows, the indices of the derivatives *not absent* and those of the variables *present* form a set of indices 1, 2, $\cdots$, n. Hence no transpositions are necessary to bring the ith variable to the ith place in C,—for principal minors $\lambda = 0$. In particular the sign-factors of the principal minors $(-1)^{\lambda}\cdot E(\because)$ of $E()$ are all $+1$.

The properties of the matrix of the hessian minors immediately yield a generalization of the empirical rule stated in the introduction. Let it be given that the hessian $E()$ and all its principal minors are positive. Then any element of the rows of the principal minors (of all orders) of the $F_m()$, since it is equal to a ratio of two elements of the column of principal minors of $E()$, is equal to a positive quantity. Hence any principal minor of the $F_m()$ has the sign of its sign-factor, and so is positive or negative according as its $\lambda + \mu$ is even or odd. But $\lambda = 0$ for principal minors. So any principal minor of $F_m()$ is negative when and only when μ is odd. Now μ is the number of the variables $x_1, x_2, \cdots, x_m$ appearing in the corresponding combination C, and can have any of the values 0, 1, 2, $\cdots$, m. Any variable x_i of the set appears in C when and only when the corresponding numeral i is missing from the upper row of indices in the symbol $F_m(\because)$ for the minor in question. When none are missing $\mu = 0$. So the minor is negative when and only when any odd number of the first m row-and-column pairs is not deleted; i.e., when the elements of the array of the minor include derivatives with regard to an odd number of the variables $p_1, p_2, \cdots, p_m$. Further, when the function e is homogeneous of degree one, its first derivatives are homogeneous functions of degree zero, wherefore the jacobian $E()$ of these derivatives vanishes identically. This causes the column of the minors of $F_n()$ to disappear from the matrix, and it causes all the minors in the row of the element $E()$ to vanish. These minors are the determinants of the second derivatives with regard to all the x's that occur in the respective functions f_m. The results thus obtained constitute a generalization of the empirical rule for the signs of the principal minors of the hessians $F_m()$. And it may be added that the elements of any row of secondary minors (including their sign-factors) in the matrix have the same sign, and vanish together.

Further cases of particular interest arise when one or more of the principal minors of $E()$ vanish identically. When m has a particular value c, then the $(c+1)$th column of the matrix is the column of the minors of $F_c()$, and the factor a_c for this column is the reciprocal of the determinant

$$(1) \qquad E\binom{c+1, \cdots, n}{c+1, \cdots, n}.$$

If this determinant vanishes identically, which it will do if the primitive function $e(x_1, x_2, \cdots, x_n)$ is homogeneous of degree one in $x_1, x_2, \cdots, x_c$, then the transformation

$$x_1' = p_1, \; \cdots, \; x_c' = p_c, \qquad x_{c+1}' = x_{c+1}, \; \cdots, \; x_n' = x_n,$$

$$f_c = e - \sum_{i=1}^{c} p_i x_i$$

is degenerate, wherefore the $(c + 1)$th column disappears from the matrix, and we have that the elements of the row of (1), i.e., the elements of the row of the combination

$$C = p_1 p_2 \cdots p_c x_{c+1} x_{c+2} \cdots x_n$$

vanish identically. The special case $c = n$ was considered in the preceding paragraph, where the condition $E() = 0$ appeared as a consequence of the circumstance that the primitive function e was a homogeneous function of degree one.

4. Special Cases of the Matrix. When $n = 1$ we have

$$de = pdx, \qquad f = e - px, \qquad df = -xdp.$$

The $2n$ elements are p, x, the two combinations C are p and x, and we have $\lambda = \mu = 0$ for all entries. The matrix asserts merely that

$$\frac{1}{d^2e/dx^2} = \frac{-d^2f/dp^2}{1},$$

i.e., that $(dp/dx)^{-1} = dx/dp$.

When $n = 2$, the $2n$ elements are p_1, x_1, p_2, x_2. Writing the six combinations C in a column, entering the values of the λ and μ, and arranging the jacobians of the letters of each combination with regard to the variables of the successive sets of independent variables, we obtain the following tabulation.

C	λ	$\frac{\partial(C\quad)}{\partial(x_1, x_2)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, x_2)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, p_2)}$
p_1x_1	1	$-E\binom{2}{1}$	0	1	$-F_1\binom{2}{1}$	0	1	$-F_2\binom{2}{1}$
p_1p_2	0	$+E(\,)$	0	0	$+F_1\binom{1}{1}$	0	0	$+F_2\binom{12}{12}$
p_1x_2	0	$+E\binom{2}{2}$	0	0	$+F_1\binom{12}{12}$	0	1	$-F_2\binom{1}{1}$
x_1p_2	0	$+E\binom{1}{1}$	0	1	$-F_1(\,)$	0	1	$-F_2\binom{2}{2}$
x_1x_2	0	$+E\binom{12}{12}$	0	1	$-F_1\binom{2}{2}$	0	2	$+F_2(\,)$
p_2x_2	0	$+E\binom{1}{2}$	0	0	$+F_1\binom{1}{2}$	1	1	$+F_2\binom{1}{2}$

The tabulated minors of order zero are equal to unity. For when the independent variables are v, w, and $C = vw$, we have

$$U\left({}^{12}_{12}\right) = \partial(v, w)/\partial(v, w) = +1 \qquad (U = E, F_m).$$

So, in more conventional notation, writing Δ, Δ_m for the hessians of e, f_m, the above matrix becomes:

$$\left\Vert \begin{array}{ccc} -\dfrac{\partial^2 e}{\partial x_1 \partial x_2} & -\dfrac{\partial^2 f_1}{\partial p_1 \partial x_2} & -\dfrac{\partial^2 f_2}{\partial p_1 \partial p_2} \\ \Delta & \dfrac{\partial^2 f_1}{\partial x_2^2} & 1 \\ \dfrac{\partial^2 e}{\partial x_1^2} & 1 & -\dfrac{\partial^2 f_2}{\partial p_2^2} \\ \dfrac{\partial^2 e}{\partial x_2^2} & -\Delta_1 & -\dfrac{\partial^2 f_2}{\partial p_1^2} \\ 1 & -\dfrac{\partial^2 f_1}{\partial p_1^2} & \Delta_2 \\ \dfrac{\partial^2 e}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f_1}{\partial x_2 \partial p_1} & \dfrac{\partial^2 f_2}{\partial p_2 \partial p_1} \end{array} \right\Vert$$

In this matrix each element of the second column differs from the corresponding element of the first by the factor $(\partial^2 e/\partial x_1^2)^{-1}$, and each element of the third column differs from the corresponding element of the first by the factor Δ^{-1}. We thus have,

$$\frac{1}{\partial^2 e/\partial x_1^2} = \frac{-\partial^2 f_1/\partial p_1^2}{1} = \frac{\partial^2 f_1/\partial x_2^2}{\Delta} = \frac{-\Delta_1}{\partial^2 e/\partial x_2^2} = \frac{\partial^2 f_1/\partial p_1 \partial x_2}{\partial^2 e/\partial x_1 \partial x_2},$$

$$\frac{1}{\Delta} = \frac{-\partial^2 f_2/\partial p_1^2}{\partial^2 e/\partial x_2^2} = \frac{-\partial^2 f_2/\partial p_2^2}{\partial^2 e/\partial x_1^2} = \frac{\Delta_2}{1} = \frac{\partial^2 f_2/\partial p_1 \partial p_2}{\partial^2 e/\partial x_1 \partial x_2}.$$

If the hessian Δ and its principal minors are positive, it follows, in accordance with the empirical rule, that

$$\partial^2 f_1/\partial p_1^2 < 0, \qquad \partial^2 f_1/\partial x_2^2 > 0, \qquad \Delta_1 < 0;$$

$$\partial^2 f_2/\partial p_1^2 < 0, \qquad \partial^2 f_2/\partial p_2^2 < 0, \qquad \Delta_2 > 0.$$

It follows also that the secondary minors,

$$\partial^2 e/\partial x_1 \partial x_2, \qquad \partial^2 f_1/\partial p_1 \partial x_2, \qquad \partial^2 f_2/\partial p_1 \partial p_2,$$

either have the same sign or vanish together.

If Δ vanishes, while its principal minors are positive, the function $f_2(p_1, p_2)$ disappears from the set of functions f_m, and we have

$$\partial^2 f_1/\partial p_1^2 < 0, \qquad \partial^2 f_1/\partial x_2^2 = 0, \qquad \Delta_1 < 0,$$

again according to rule. Here also $\partial^2 e/\partial x_1 \partial x_2$ and $\partial^2 f_1/\partial p_1 \partial x_2$ either have the same sign or vanish together.

C	λ	$\frac{\partial(C\quad)}{\partial(x_1, x_2, x_3)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, x_2, x_3)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, p_2, x_3)}$	λ	μ	$\frac{\partial(C\quad)}{\partial(p_1, p_2, p_3)}$
$p_1x_1p_2$	1	$-E\binom{3}{1}$	0	1	$-F_1\binom{3}{1}$	1	1	$+F_2\binom{23}{12}$	1	1	$+F_3\binom{23}{12}$
$p_1x_1x_2$	2	$+E\binom{23}{12}$	1	1	$+F_1\binom{23}{12}$	0	2	$+F_2\binom{3}{1}$	0	2	$+F_3\binom{3}{1}$
$p_1x_1p_3$	1	$-E\binom{2}{1}$	0	1	$-F_1\binom{2}{1}$	0	1	$-F_2\binom{2}{1}$	0	1	$-F_3\binom{23}{13}$
$p_1x_1x_3$	1	$-E\binom{23}{13}$	0	1	$-F_1\binom{23}{13}$	0	1	$-F_2\binom{23}{13}$	0	2	$+F_3\binom{2}{1}$
$p_1p_2x_2$	1	$-E\binom{3}{2}$	1	0	$-F_1\binom{13}{12}$	0	1	$-F_2\binom{13}{12}$	0	1	$-F_3\binom{13}{12}$
$p_1p_2p_3$	0	$+E(\ )$	0	0	$+F_1\binom{1}{1}$	0	0	$+F_2\binom{12}{12}$	0	0	$+F_3\binom{123}{123}$
$p_1p_2x_3$	0	$+E\binom{3}{3}$	0	0	$+F_1\binom{13}{13}$	0	0	$+F_2\binom{123}{123}$	0	1	$-F_3\binom{12}{12}$
$p_1x_2p_3$	0	$+E\binom{2}{2}$	0	0	$+F_1\binom{12}{12}$	0	1	$-F_2\binom{1}{1}$	0	1	$-F_3\binom{13}{13}$
$p_1x_2x_3$	0	$+E\binom{23}{23}$	0	0	$+F_1\binom{123}{123}$	0	1	$-F_2\binom{13}{13}$	0	2	$+F_3\binom{1}{1}$
$*p_1p_3x_3$	0	$+E\binom{2}{3}$	0	0	$+F_1\binom{12}{13}$	0	0	$+F_2\binom{12}{13}$	1	1	$+F_3\binom{12}{13}$
$x_1p_2x_2$	1	$-E\binom{13}{12}$	1	1	$+F_1\binom{3}{2}$	0	2	$+F_2\binom{3}{2}$	0	2	$+F_3\binom{3}{2}$
$x_1p_2p_3$	0	$+E\binom{1}{1}$	0	1	$-F_1(\ )$	0	1	$-F_2\binom{2}{2}$	0	1	$-F_3\binom{23}{23}$
$x_1p_2x_3$	0	$+E\binom{13}{13}$	0	1	$-F_1\binom{3}{3}$	0	1	$-F_2\binom{23}{23}$	0	2	$+F_3\binom{2}{2}$
$x_1x_2p_3$	0	$+E\binom{12}{12}$	0	1	$-F_1\binom{2}{2}$	0	2	$+F_2(\ )$	0	2	$+F_3\binom{3}{3}$
$x_1x_2x_3$	0	$+E\binom{123}{123}$	0	1	$-F_1\binom{23}{23}$	0	2	$+F_2\binom{3}{3}$	0	3	$-F_3(\ )$
$*x_1p_3x_3$	0	$+E\binom{12}{13}$	0	1	$-F_1\binom{2}{3}$	0	1	$-F_2\binom{2}{3}$	1	2	$-F_3\binom{2}{3}$
$*p_2x_2p_3$	0	$+E\binom{1}{2}$	0	0	$+F_1\binom{1}{2}$	1	1	$+F_2\binom{1}{2}$	1	1	$+F_3\binom{13}{23}$
$*p_2x_2x_3$	0	$+E\binom{13}{23}$	0	0	$+F_1\binom{13}{23}$	1	1	$+F_2\binom{13}{23}$	1	2	$-F_3\binom{1}{2}$
$*p_2p_3x_3$	0	$+E\binom{1}{3}$	0	0	$+F_1\binom{1}{3}$	1	0	$-F_2\binom{12}{23}$	2	1	$-F_3\binom{12}{23}$
$*x_2p_3x_3$	1	$-E\binom{12}{23}$	1	0	$-F_1\binom{12}{23}$	0	1	$-F_2\binom{1}{3}$	1	2	$-F_3\binom{1}{3}$

It may be remarked that a recently described classification* of the elements of a certain set of derivatives, which is of consequence in thermodynamics, is essentially an application of the properties of the above matrix for $n = 2$.

When $n = 3$ the $2n$ elements are p_1, x_1, p_2, x_2, p_3, x_3, and the matrix of 20×4 elements is as on page 82.

The factors that convert the elements of the first column of this matrix into those of the second, the third, and the fourth are the reciprocals of the principal minors

$$E\binom{23}{23}, \qquad E\binom{3}{3}, \qquad E(\ ).$$

We thus have, arranging principal minors and secondary minors on separate lines, and omitting duplicate secondary minors (of the starred rows in the tabulation),

$$\frac{F_1\binom{1}{1}}{E(\ )} = \frac{F_1\binom{13}{13}}{E\binom{3}{3}} = \frac{F_1\binom{12}{12}}{E\binom{2}{2}} = \frac{1}{E\binom{23}{23}} = \frac{-F_1(\ )}{E\binom{1}{1}} = \frac{-F_1\binom{3}{3}}{E\binom{13}{13}} = \frac{-F_1\binom{2}{2}}{E\binom{12}{12}} = \frac{-F_1\binom{23}{23}}{1}$$

$$= \frac{F_1\binom{3}{1}}{E\binom{3}{1}} = \frac{F_1\binom{23}{12}}{E\binom{23}{12}} = \frac{F_1\binom{2}{1}}{E\binom{2}{1}} = \frac{F_1\binom{23}{13}}{E\binom{23}{13}} = \frac{F_1\binom{13}{12}}{E\binom{3}{2}} = \frac{-F_1\binom{3}{2}}{E\binom{13}{12}},$$

together with a similar set of equations for the minors of $F_2()$, and a third set for the minors of $F_3()$.

If the hessian $E()$ and its principal minors are positive, it follows, in accordance with the empirical rule, that the principal minors of the $F_m()$ have the signs prefixed to them in the table. It follows also that the sign of any secondary minor of any $F_m()$ is determined by the sign of the secondary minor of $E()$ in the same row, and that all the minors of the row vanish together.

If $E()$ vanishes, while its principal minors are positive, the function f_3 disappears. But the signs of the minors of $F_1()$ and $F_2()$ are determined as before, save that

$$F_1\binom{1}{1} = F_2\binom{12}{12} = 0,$$

which accords with the rule.

In constructing the matrix for any value of n, the symbols $F_m(:::)$ of any row may be rapidly obtained from the symbol $E(:::)$ of the same row by the rule,—Replace E by F_m, delete all *common* indices $1, \cdots, m$, and insert all *missing* indices $m + 1, \cdots, n$.

In the case $n = 4$, where the operation of the empirical rule was first observed, the matrix of the *principal* minors is as follows.

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* J. E. Trevor, Amer. Math. Monthly, vol. 27, p. 258 (1920).

C	λ	$\frac{\partial(C\qquad)}{\partial(x_1, x_2, x_3, x_4)}$	λ	μ	$\frac{\partial(C\qquad)}{\partial(p_1, x_2, x_3, x_4)}$	λ	μ	$\frac{\partial(C\qquad)}{\partial(p_1, p_2, x_3, x_4)}$	λ	μ	$\frac{\partial(C\qquad)}{\partial(p_1, p_2, p_3, x_4)}$	λ	μ	$\frac{\partial(C\qquad)}{\partial(p_1, p_2, p_3, p_4)}$
$x_1x_2x_3x_4$	0	$+E\binom{1234}{1234}$	0	1	$-F_1\binom{234}{234}$	0	2	$+F_2\binom{34}{34}$	0	3	$-F_3\binom{4}{4}$	0	4	$+F_4(\;)$
$p_1x_2x_3x_4$	0	$+E\binom{234}{234}$	0	0	$+F_1\binom{1234}{1234}$	0	1	$-F_2\binom{134}{134}$	0	2	$+F_3\binom{14}{14}$	0	3	$-F_4\binom{1}{1}$
$x_1p_2x_3x_4$	0	$+E\binom{134}{134}$	0	1	$-F_1\binom{34}{34}$	0	1	$-F_2\binom{234}{234}$	0	2	$+F_3\binom{24}{24}$	0	3	$-F_4\binom{2}{2}$
$x_1x_2p_3x_4$	0	$+E\binom{124}{124}$	0	1	$-F_1\binom{24}{24}$	0	2	$+F_2\binom{4}{4}$	0	2	$+F_3\binom{34}{34}$	0	3	$-F_4\binom{3}{3}$
$x_1x_2x_3p_4$	0	$+E\binom{123}{123}$	0	1	$-F_1\binom{23}{23}$	0	2	$+F_2\binom{3}{3}$	0	3	$-F_3(\;)$	0	3	$-F_4\binom{4}{4}$
$p_1p_2x_3x_4$	0	$+E\binom{34}{34}$	0	0	$+F_1\binom{134}{134}$	0	0	$+F_2\binom{1234}{1234}$	0	1	$-F_3\binom{124}{124}$	0	2	$+F_4\binom{12}{12}$
$p_1x_2p_3x_4$	0	$+E\binom{24}{24}$	0	0	$+F_1\binom{124}{124}$	0	1	$-F_2\binom{14}{14}$	0	1	$-F_3\binom{134}{134}$	0	2	$+F_4\binom{13}{13}$
$x_1p_2p_3x_4$	0	$+E\binom{14}{14}$	0	1	$-F_1\binom{4}{4}$	0	1	$-F_2\binom{24}{24}$	0	1	$-F_3\binom{234}{234}$	0	2	$+F_4\binom{23}{23}$
$p_1x_2x_3p_4$	0	$+E\binom{23}{23}$	0	0	$+F_1\binom{123}{123}$	0	1	$-F_2\binom{13}{13}$	0	2	$+F_3\binom{1}{1}$	0	2	$+F_4\binom{14}{14}$
$x_1p_2x_3p_4$	0	$+E\binom{13}{13}$	0	1	$-F_1\binom{3}{3}$	0	1	$-F_2\binom{23}{23}$	0	2	$+F_3\binom{2}{2}$	0	2	$+F_4\binom{24}{24}$
$x_1x_2p_3p_4$	0	$+E\binom{12}{12}$	0	1	$-F_1\binom{2}{2}$	0	2	$+F_2(\;)$	0	2	$+F_3\binom{3}{3}$	0	2	$+F_4\binom{34}{34}$
$p_1p_2p_3x_4$	0	$+E\binom{4}{4}$	0	0	$+F_1\binom{14}{14}$	0	0	$+F_2\binom{124}{124}$	0	0	$+F_3\binom{1234}{1234}$	0	1	$-F_4\binom{123}{123}$
$p_1p_2x_3p_4$	0	$+E\binom{3}{3}$	0	0	$+F_1\binom{13}{13}$	0	0	$+F_2\binom{123}{123}$	0	1	$-F_3\binom{12}{12}$	0	1	$-F_4\binom{124}{124}$
$p_1x_2p_3p_4$	0	$+E\binom{2}{2}$	0	0	$+F_1\binom{12}{12}$	0	1	$-F_2\binom{1}{1}$	0	1	$-F_3\binom{13}{13}$	0	1	$-F_4\binom{134}{134}$
$x_1p_2p_3p_4$	0	$+E\binom{1}{1}$	0	1	$-F_1(\;)$	0	1	$-F_2\binom{2}{2}$	0	1	$-F_3\binom{23}{23}$	0	1	$-F_4\binom{234}{234}$
$p_1p_2p_3p_4$	0	$+E(\;)$	0	0	$+F_1\binom{1}{1}$	0	0	$+F_2\binom{12}{12}$	0	0	$+F_3\binom{123}{123}$	0	0	$+F_4\binom{1234}{1234}$

When $E()$ and its principal minors are positive we have that eight of the principal minors of each of the $F_m()$ are negative, while the other eight are positive. If $E()$ vanishes, while its principal minors remain positive, the last column disappears from the table, and the signs are distributed as before save that

$$F_1\binom{1}{1} = F_2\binom{12}{12} = F_3\binom{123}{123} = 0;$$

all of which accords with the rule. The whole matrix of 350 elements is too extensive to quote.

Cornell University,
June, 1920.